Order and Disorder in Multiscale Substitution Tilings

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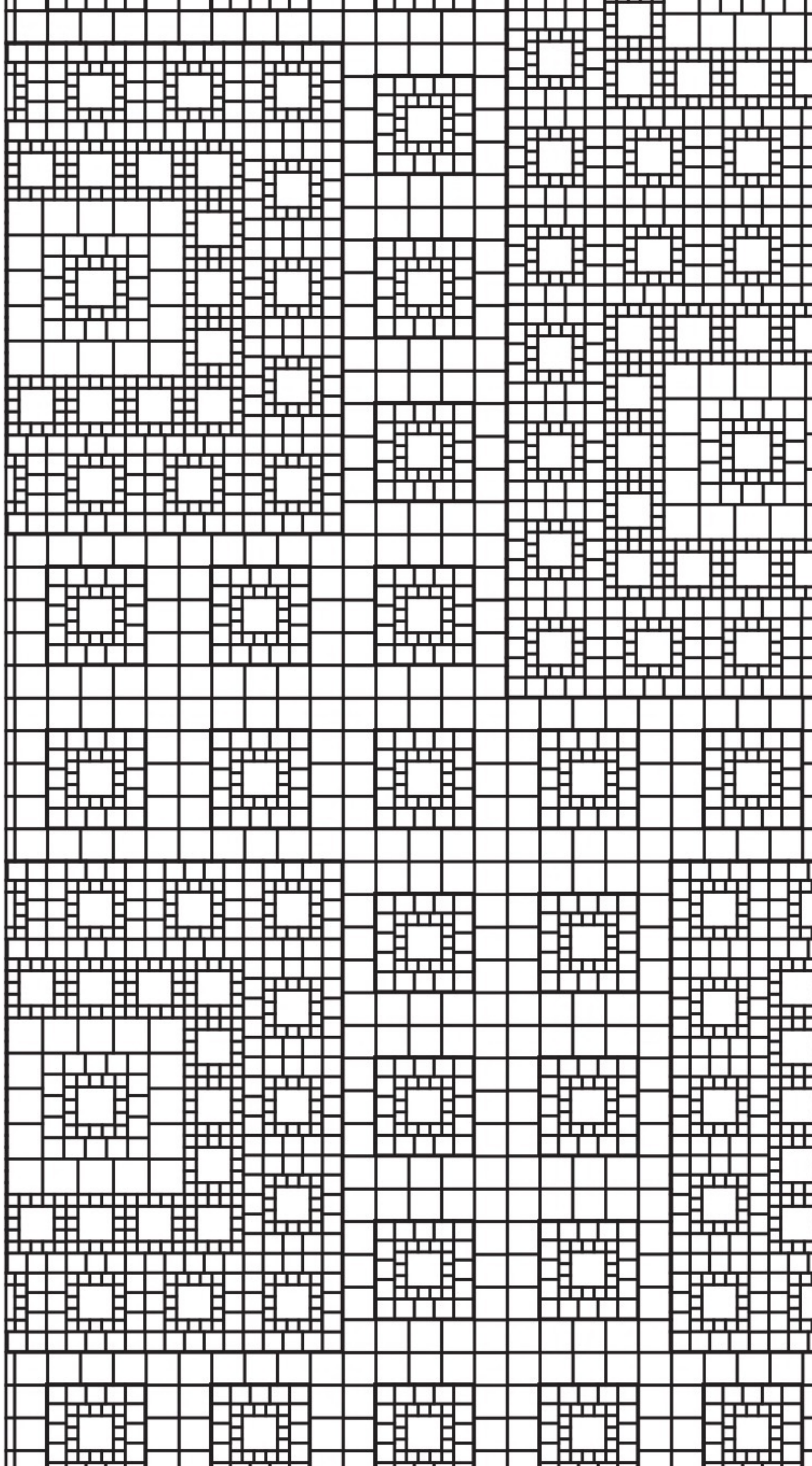
Tel Aviv Analysis Seminar, 2021

Partially based on joint work with Yaar Solomon

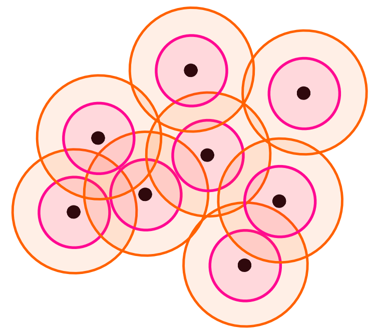
Plan of Talk

- Introduction
- Bounded displacement and bilipschitz equivalence
- Multiscale substitution tilings



Delone Sets

A uniformly discrete and relatively dense set $\Lambda \subseteq \mathbb{R}^d$ is called Delone.



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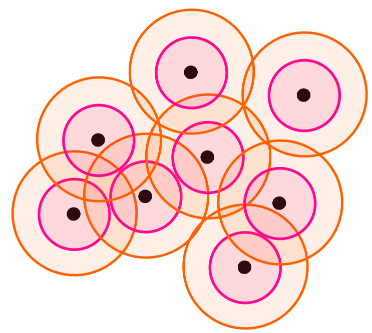
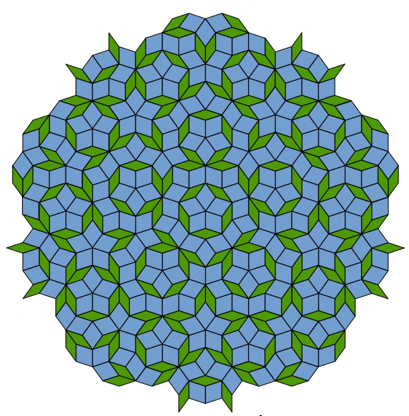
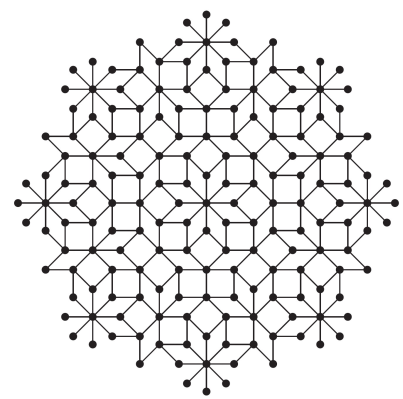


Examples Lattices, sets induced by tilings and cut-and-project sets



From Wikipedia



From Baake and Grimm's Aperiodic Order Vol 1

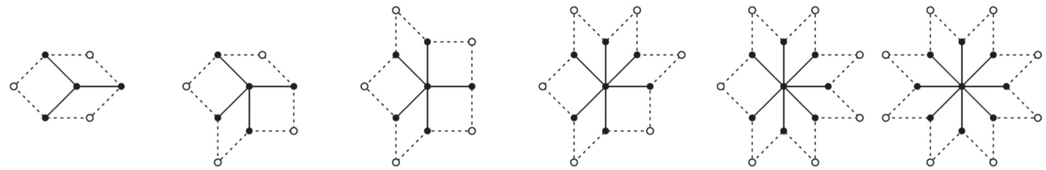
A basic problem is to classify and measure how ordered or disordered a given Delone set is, compared to a lattice.

Lattice-like Properties

For $x \in \Lambda$, $r > 0$ the r-patch of Λ at x is $P_{\Lambda,r}(x) = (\Lambda - x) \cap B(0,r)$

- Finite local complexity (FLC)

  $\forall\, r > 0 \quad \#\{P_{\Lambda,r}(x) \mid x \in \Lambda\} < \infty$



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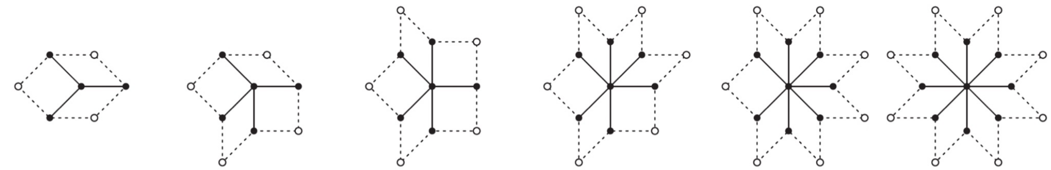

From Baake and Grimm's Aperiodic Order Vol 1

"Face it, Fred—you're lost."

- Repetitivity $\forall\, r > 0 \;\exists\, R = R(r)$ so that every R-ball contains a copy of every r-patch. Linear repetitivity if $R(r)$ is linear. Uniform patch frequency if patches appear in well-defined frequencies.

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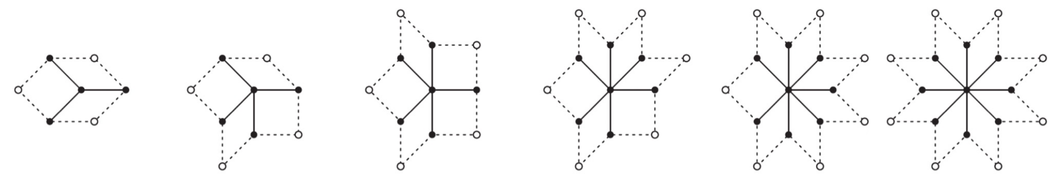

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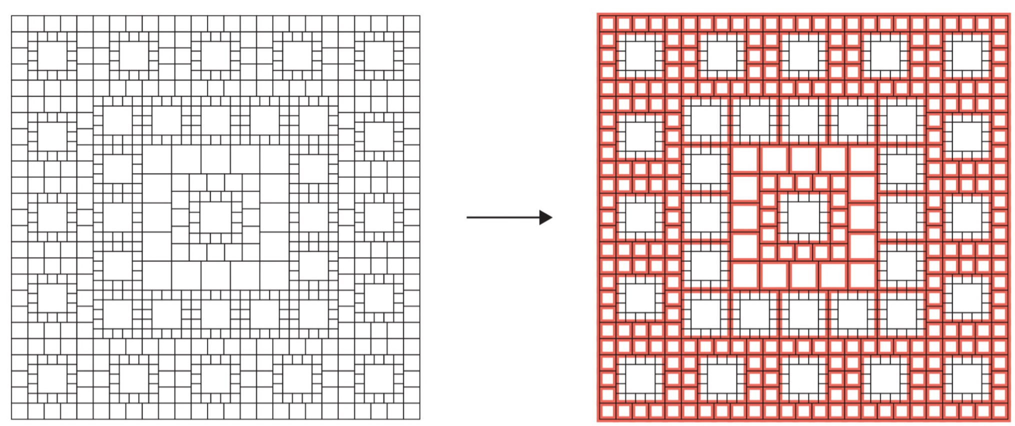
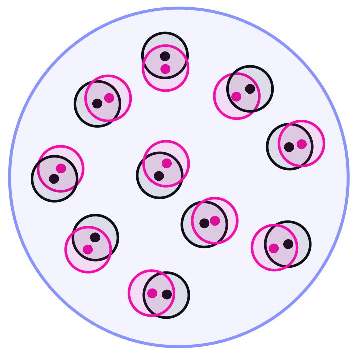
- Self-similarity there exists $\alpha>1$ so that $\alpha\Lambda \subset \Lambda$



Spaces and Dynamical Systems of Delone Sets

Set $X_\Lambda = \overline{\{\Lambda + t \mid t \in \mathbb{R}^d\}}$, where the closure is with respect to a natural topology on Delone sets (induced by the Hausdorff metric restricted to centered balls)



- Λ is (almost) repetitive $\Longleftrightarrow$ The dynamical system $(X_\Lambda, \mathbb{R}^d)$ is minimal (every orbit is dense)

- (almost) linear repetitivity $\Longrightarrow$ unique ergodicity (unique invariant measure)

(Radin Wolff '92, Solomyak '97, Damanik Lenz '01, Lagarias Pleasants '03, Frettlöh Richard '14)

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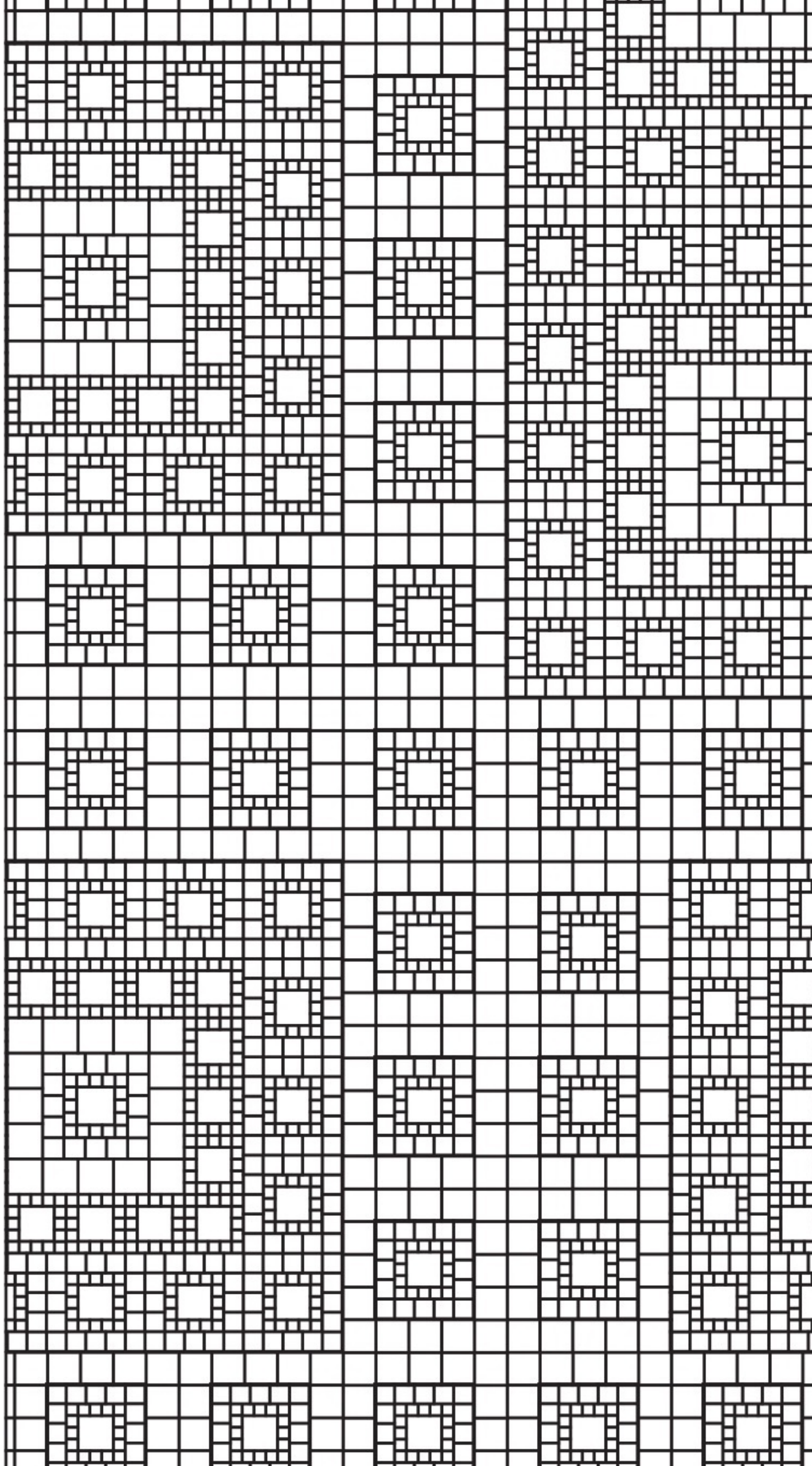
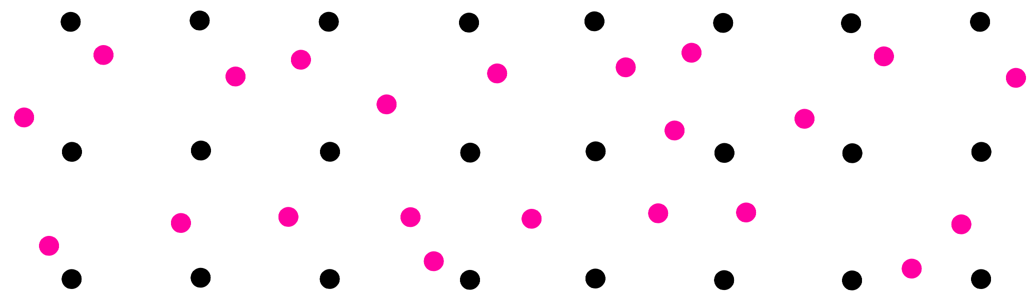


BD and BL Equivalence Relations

- Delone sets $\Lambda, \Gamma \subseteq \mathbb{R}^d$ are bounded displacement (BD) equivalent if $\exists$ bijection $\varphi: \Lambda \to \Gamma$ that moves every point a bounded distance.



- Λ, Γ are bilipschitz (BL) equivalent if $\exists$ $L>0$ and bijection $\varphi: \Lambda \to \Gamma$

$$\forall x, y \in \Lambda \qquad \frac{1}{L} \leqslant \frac{\|\varphi(x) - \varphi(y)\|}{\|x-y\|} \leqslant L$$

- Λ is uniformly spread if it is BD to $\alpha\mathbb{Z}^d$ for some $\alpha > 0$, and rectifiable if it is BL to $\mathbb{Z}^d$.

BD and BL Equivalence Relations

- Not all Delone sets are rectifiable (Burago Kleiner, McMullen '98, Cortez Navas '16)
- Linear repetitivity $\Rightarrow$ rectifiability (Aliste-Prieto, Coronel, Gambaudo '13)

(use Burago-Kleiner '02 sufficient condition for rectifiability)

Theorem (SS3 ≥ '21) True also for almost linear repetitivity.

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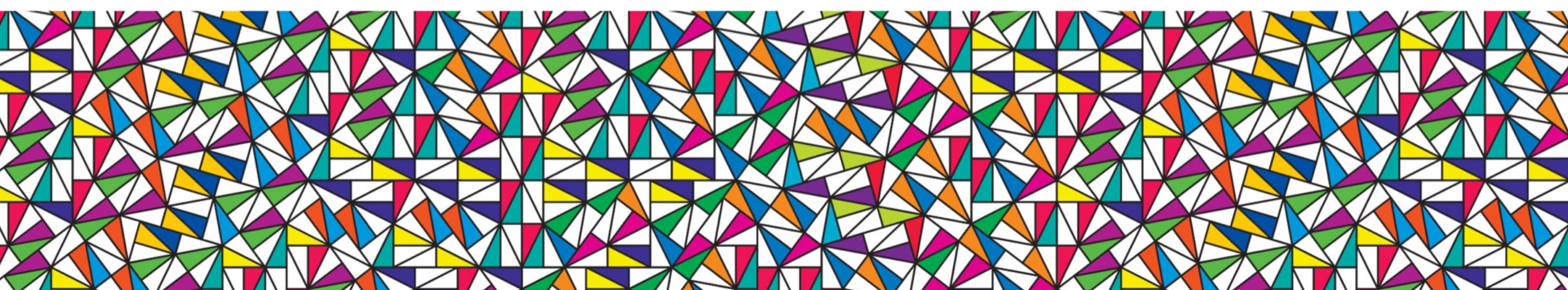
- Sets associated with tilings with a single tile are uniformly spread

$\Rightarrow$ lattices & periodic sets (Duneau, Oguey '90, Hall's marriage theorem)



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Theorem (SS3≥'21) True also for almost linear repetitivity.

Laczkovich '92 For a Delone set $\Lambda \subset \mathbb{R}^d$ the following are equivalent:

- Λ is uniformly spread
- There exist $\alpha, C > 0$ so that $\forall\ A \in \mathcal{Q}_d = \{\text{finite unions of lattice cubes}\}$

discrepancy $\quad |\#(A \cap \Lambda) - \alpha \cdot \mathrm{vol}(A)| \leq C \cdot \mathrm{vol}_{d-1}(\partial A)$

BD Equivalence Criterion and Dichotomy

Theorem (FSS '21 and SS2 '21) The following are equivalent:

- Λ and Γ are not BD equivalent
- There exists a sequence of sets $A_m \in Q_d$ so that

$$\frac{|\#(A_m \cap \Lambda) - \#(A_m \cap \Gamma)|}{\mathrm{vol}_{d-1}(\partial A_m)} \xrightarrow{m\to\infty} \infty$$

Theorem (SS2 '21) Let X be a minimal space of Delone sets.

- Either $\exists\, \Lambda \in X$ uniformly spread, and then every $\Lambda \in X$ is such.
- Or X contains continuously many distinct BD class representatives.

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$\Rightarrow$ Substitution tilings (Solomon '14)

$\Rightarrow$ Cut-and-project sets (Haynes, Kelly, Weiss '14, Frettlöh, Garber '18)

$\Rightarrow$ Incommensurable multiscale tilings (SS1 '21)

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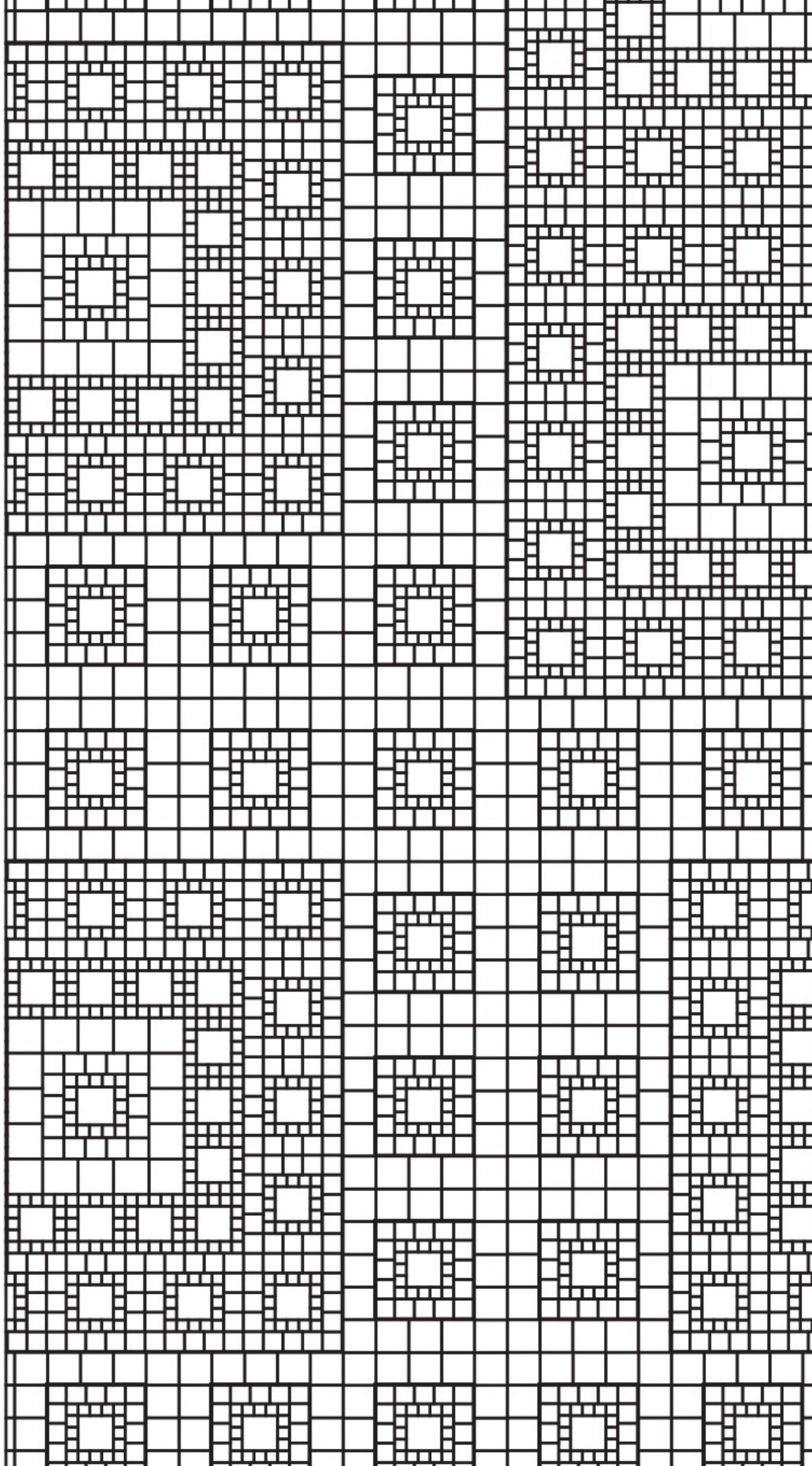
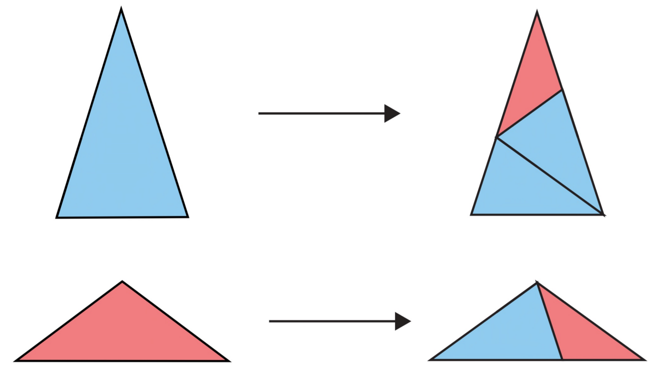


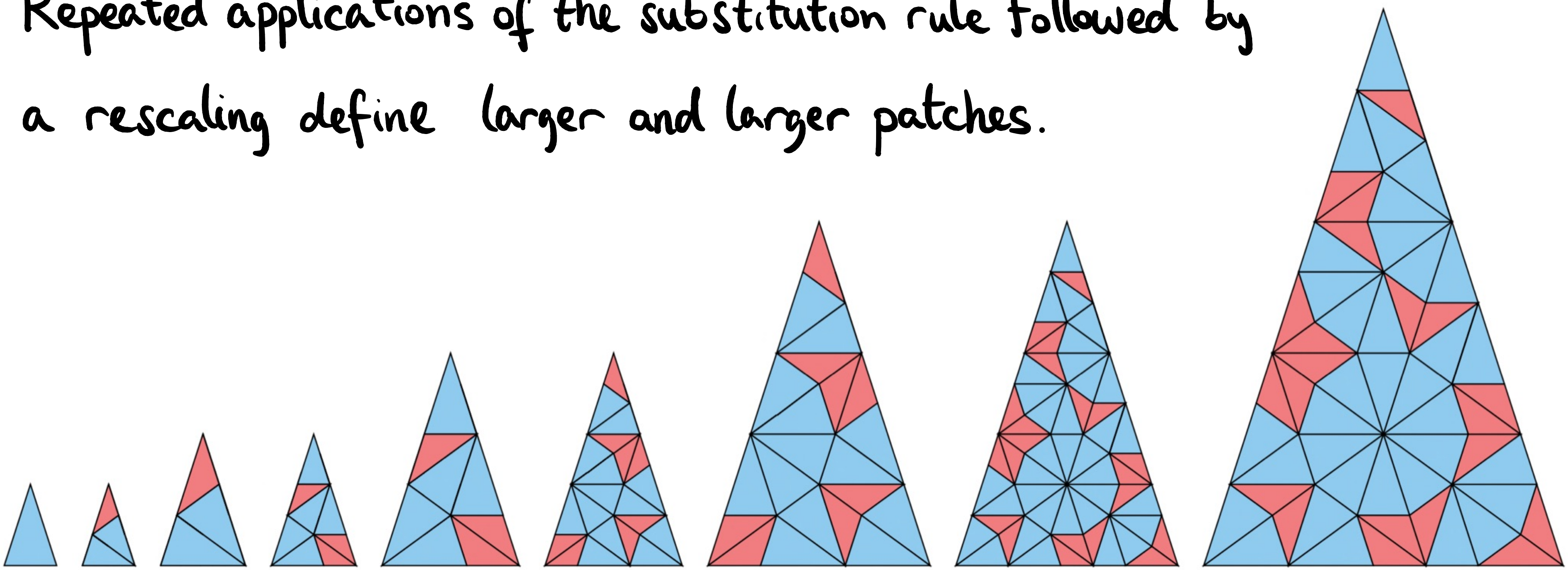
Substitution Tilings

A tiling is a collection of tiles with disjoint interiors that covers $\mathbb{R}^d$.

A substitution rule on a set of prototiles is a tessellation of each prototile by rescaled prototiles, with a fixed scale $\in (0,1)$

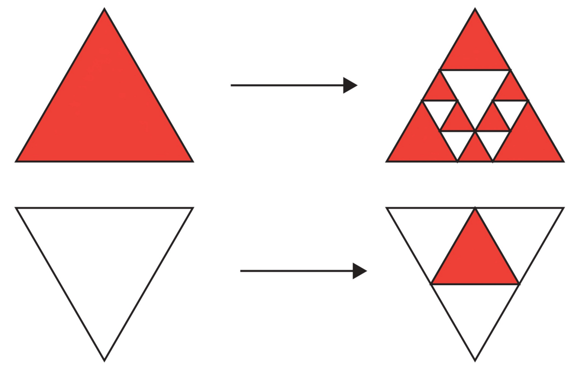


Repeated applications of the substitution rule followed by a rescaling define larger and larger patches.

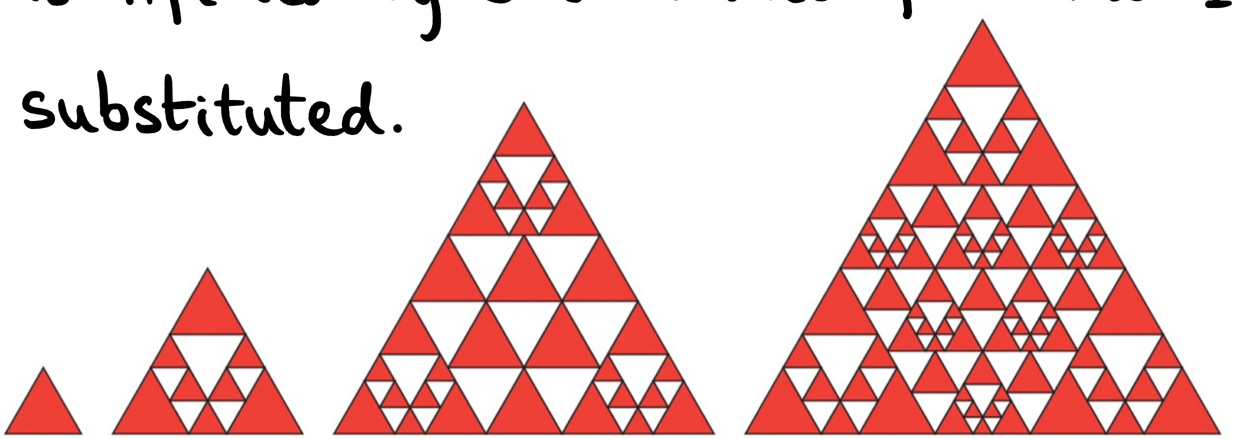


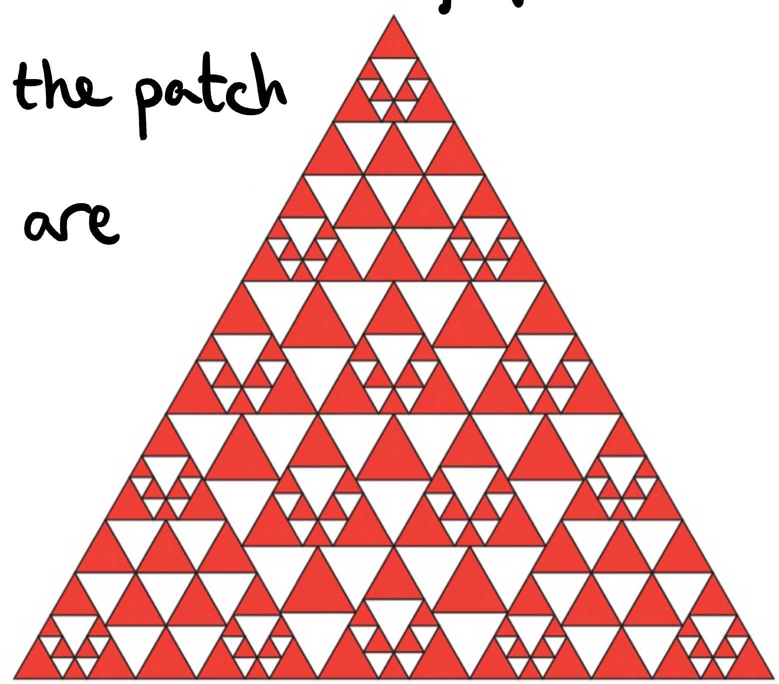
Incommensurable Multiscale Substitution Tilings

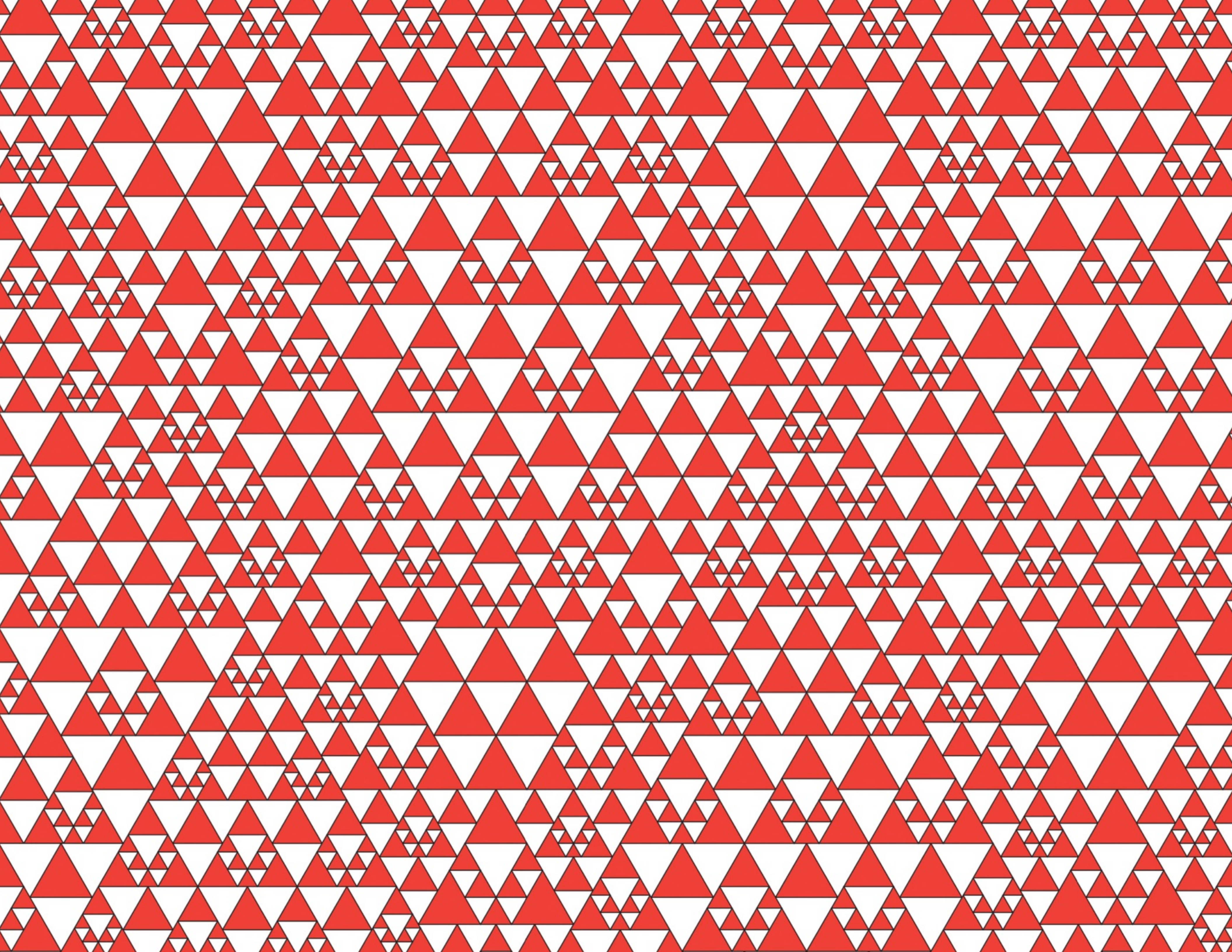
A multiscale substitution scheme σ in $\mathbb{R}^d$ consists of a substitution rule on unit volume prototiles $T_1, \ldots, T_n$, where various different scales appear and satisfy a simple incommensurabily condition.



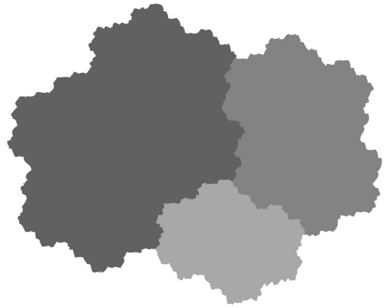
A time-dependant substitution semiflow F_t defines a family of patches: At time $t=0$ $F_t(T) = T$, and as t increases the patch is inflated by e^t and tiles of volume > 1 are substituted.

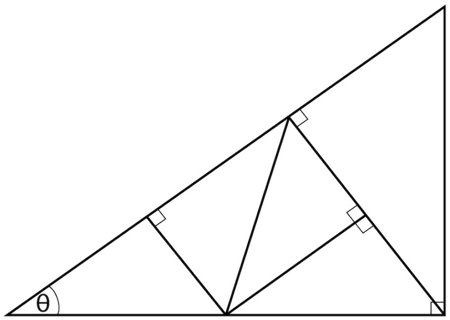






Some Predecessors

- Rauzy's fractal '81  multiple (but commensurable) scales



- Conway and Radin's pinwheel tiling '94

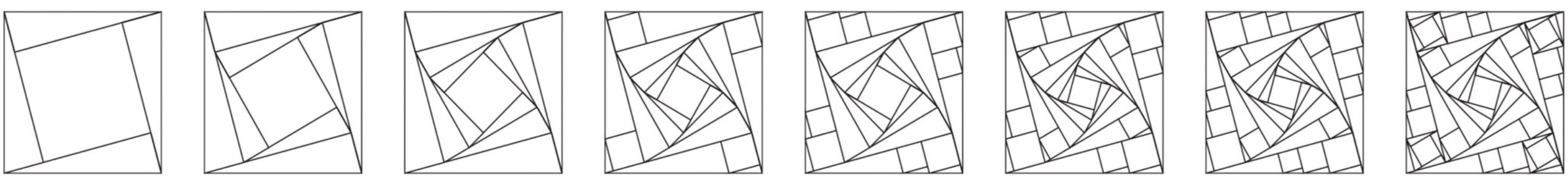
$\theta = \arctan 1/2 \Rightarrow$ same triangle incommensurable directions

- Sadun's generalized pinwheel tilings '98

- α-Kakutani sequences in $[0,1]$ '76 

always split longest interval

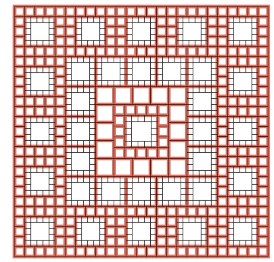
- S '20: multiscale substitution Kakutani sequences of partitions



Incommensurable Multiscale Substitution Tilings

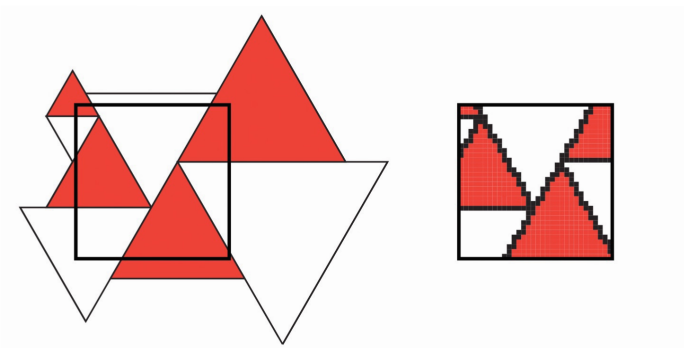
Theorem (SS1 '21) Let T be an incommensurable tiling.

- Tiles and patches appear in a dense set of scales $\Rightarrow$ not FLC
- Periodic orbits of F_t $\Rightarrow$ self similar tilings



- T has uniform patch frequencies
- $(X_T, \mathbb{R}^d)$ is minimal (T is almost repetitive) and uniquely ergodic

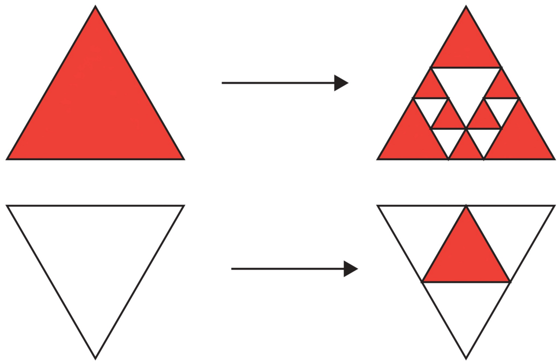
Proof follows Lee-Solomyak's '19 "pixelization" approach

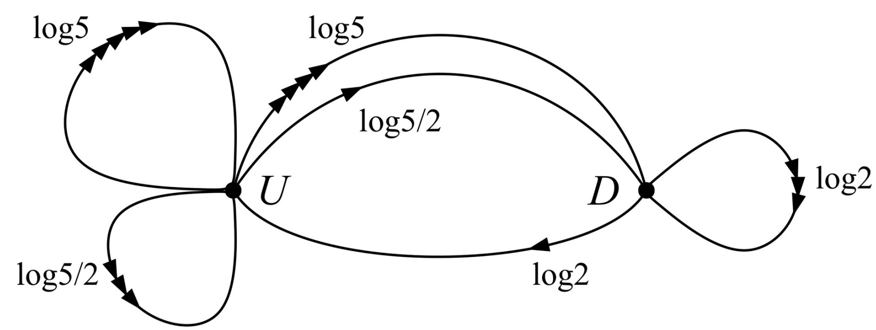


Theorem (SS3 ≥ '21) almost repetitivity is not linear

The Associated Graph G_σ

A directed weighted graph is defined according to σ





Vertices model the prototiles

Edges model the tiles appearing in the substitution rule with

Lengths $= \log(1/\text{scale})$

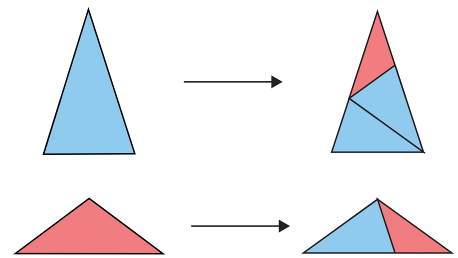
σ is incommensurable if G_σ contains two closed paths of lengths $\frac{a}{b} \notin \mathbb{Q}$.

Incommensurable multiscale substitution schemes generate a new distinct class of tilings of $\mathbb{R}^d$.

Counting in Multiscale Substitution Tilings

Substitution # tiles in patches = entries of powers of the **substitution matrix** S

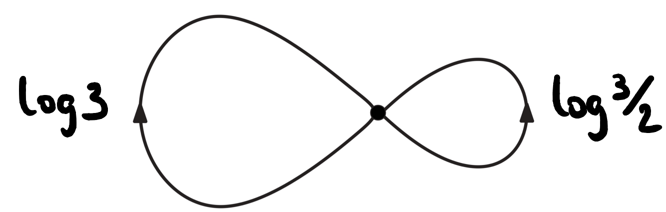
Perron-Frobenius Theorem



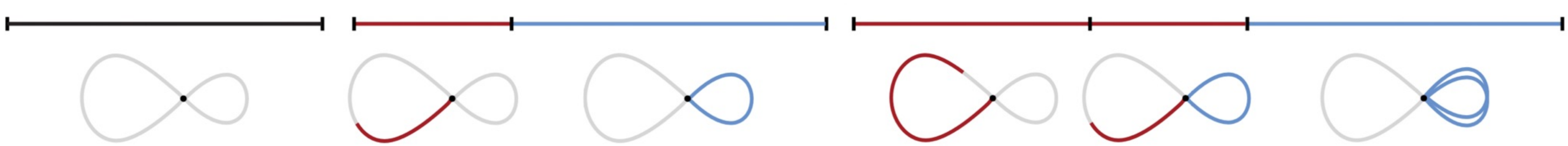
$$\Rightarrow \quad S = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Multiscale $\{\text{Tiles in } F_t(T_i)\} \longleftrightarrow \{\text{Directed walks of length } t \text{ in } G_\sigma \text{ originating at vertex } i\}$

Example the $\frac{1}{3}$-**Kakutani scheme** in $\mathbb{R}$:



the patches $F_0(I)$, $F_{\log\frac{3}{2}}(I)$, $F_{2\log\frac{3}{2}}(I)$ and their respective walks



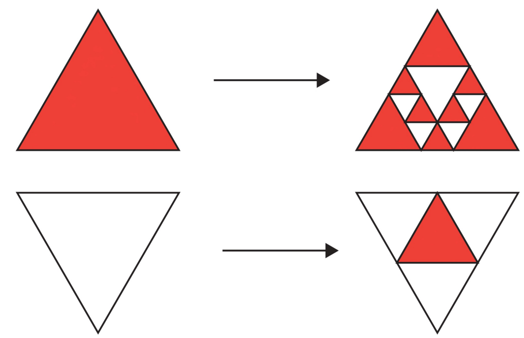
Counting in Multiscale Substitution Tilings

Theorem (SS'21, relying on Kiro, Smilansky×2 '20)

$$\#\{\text{tiles in } F_t(T)\} = \frac{v^T(S_\sigma - V_\sigma)\mathbb{1}}{v^T H_\sigma \mathbb{1}} \cdot \underbrace{e^{dt}}_{\mathrm{vol}(F_t(T))} + \text{ERROR TERM}, \quad t \to \infty$$

combinatorics matrix $\quad (S_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} 1$

\# reds in white $\quad S_\sigma = \begin{pmatrix} 8 & 5 \\ 1 & 3 \end{pmatrix}$



volume matrix $\quad (V_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} \mathrm{vol}(T)$

total red area in white $\quad V_\sigma = \begin{pmatrix} \frac{17}{25} & \frac{8}{25} \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$

entropy matrix $\quad (H_\sigma)_{ij} = \sum_{\substack{T \text{ of type } j \\ \text{in } T_i}} -\mathrm{vol}(T) \cdot \log \mathrm{vol}(T)$

contribution of reds to entropy of white

$$H_\sigma = \begin{pmatrix} -\frac{12}{25}\log\frac{4}{25} - \frac{5}{25}\log\frac{1}{25} & -\frac{4}{25}\log\frac{4}{25} - \frac{4}{25}\log\frac{1}{25} \\ -\frac{1}{4}\log\frac{1}{4} & -\frac{3}{4}\log\frac{1}{4} \end{pmatrix}$$

and v^T = left Perron-Frobenius eigenvector of V_σ

Theorem (SS'21) $\exists k \in \mathbb{N} \ \forall t_0 > 0 \ \exists t \geq t_0$: ERROR TERM $\geq C \dfrac{e^{dt}}{t^k}$

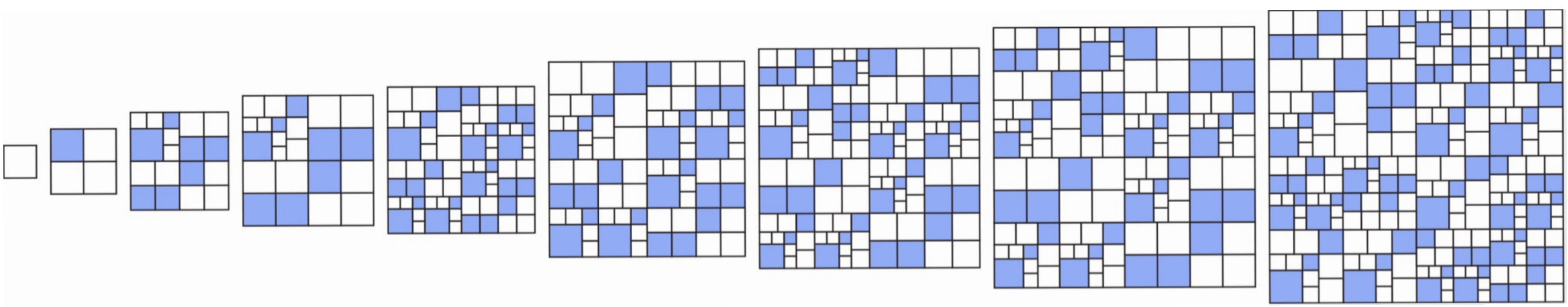
$\Rightarrow$ Incommensurable tilings are never uniformly spread.

Counting in Multiscale Substitution Tilings

Theorem (S ≥'21, relying on Kiro, Smilansky ×2 '20)

Similar asymptotic formulas for:

- $\#\{$tiles of type r and vol $\in [a,b]$ in $F_t(T)\}$
- volume$\left(\bigcup\{\text{tiles of type } r \text{ and vol} \in [a,b] \text{ in } F_t(T)\}\right)$
- Expected values for random partitions



Counting in Multiscale Substitution Tilings

Theorem (S ≥ '21, relying on Kiro, Smilansky×2 '20) similar formulas for

- **Gap distribution** Λ = Delone set of tile boundaries in a 1-dim tiling

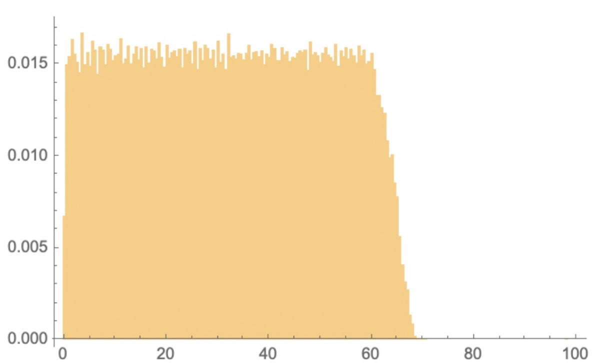
$$\frac{\#\{\text{Neighbors in } \Lambda\cap[-N,N] \text{ of distance} \in [a,b]\}}{\#\{\Lambda\cap[-N,N]\}} \longrightarrow \int_a^b \frac{v^T C_\sigma(x)\,\mathbb{1}}{v^T H_\sigma \mathbb{1}}\,dx$$

where

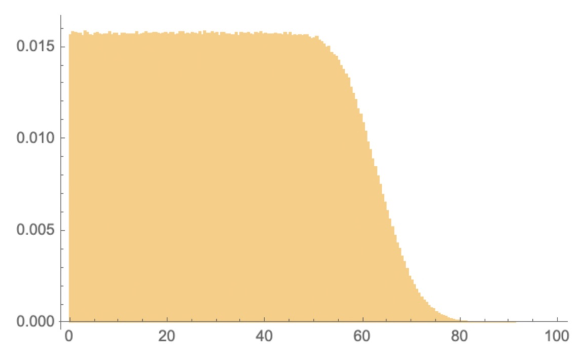
$$\left(C_\sigma(x)\right)_{ij} = \sum_{\substack{T \text{ of type } j\\ \text{in } T_i}} \begin{cases} \frac{\operatorname{vol} T}{x^2} & , \operatorname{vol} T < x \leq 1 \\ 0 & , \text{otherwise} \end{cases}$$

- **Numerics** for **pair correlations** are consistent with **Poisson process**

```
list = {0, 3^10}; i = 1;
Do[While[list[[i + 1]] - list[[i]] > 1,
  list = Insert[list, list[[i]] + (list[[i + 1]] - list[[i]]) / 3, i + 1]], {i, 91 005}];
gaps = Flatten[Table[N[Differences[list, 1, j]], {j, 1, 100}]];
Histogram[gaps, {0, 100, 0.5}, "PDF"]
```




```
averagegap = 1 / (- (1 / 3) * Log[1 / 3] - (2 / 3) * Log[2 / 3]);
list = Accumulate[RandomVariate[ExponentialDistribution[averagegap], 90 000]];
gaps = Flatten[Table[N[Differences[list, 1, j]], {j, 1, 100}]];
Histogram[gaps, {0, 100, 0.5}, "PDF"]
```



Thank You!